NON-HOMOGENEOUS LINEAR EQUATIONS IN INFINITELY MANY UNKNOWNS.

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Introduction. In this paper we consider the solution of the system of non-homogeneous linear equations in infinitely many unknowns,

$$(1) \qquad \sum_{k=1}^{\infty} a_{ik} x_k = c_i \qquad (i = 1, 2, 3, \cdots),$$

with the object of subjecting the matrix (a_{ik}) to as few restrictions as possible. We are concerned, in particular, with the derivation of conditions to be imposed on the constants $\{c_i\}$ in order that a solution $\{x_i\}$ of (1) exist, and the determination of the character of the solution.

From the articles dealing with linear equations in infinitely many unknowns we cite only those results which have some bearing on the problem stated above.*

Assuming that the matrix (a_{ik}) is such that $\Sigma_{k=1}^{\infty} a_{ik}^2$ converge for every i, E. Schmidt† derives the necessary and sufficient condition, to be imposed on the $\{c_i\}$, for the existence of a solution $\{x_i\}$ of *finite norm*. On account of the complexity of the processes involved, these conditions are difficult to apply in particular cases.

The method of Kösseritzsch‡ consists in the reduction of a general system of equations to one for which

$$(2) \qquad a_{ii} \neq 0, \qquad a_{ik} = 0, \qquad i > k,$$

and in the formal expression of the solution $\{x_i\}$ in terms of the $\{c_i\}$ by means of a reciprocal matrix. Helge von Koch§ and Carmichael‖ have

* Hilbert and Toeplitz consider the following problem: the determination of conditions to be imposed on the matrix, already restricted to be limited or even more, in order that a solution of *finite norm* exist *whenever* the constants $\{c_i\}$ are of *finite norm*. In some articles dealing with infinite determinants Helge von Koch considers a similar problem.

† Rendiconti del Circolo Matematico di Palermo, XXV, 1908, pp. 53–77. See also Bôcher and Brand, *Annals of Mathematics*, 1912, pp. 167–186.

‡ "Über die Auflösung eines Systems von unendlich vielen linearen Gleichungen," Zeitschrift für Mathematik und Physik, t. XV, 1870, pp. 1–15, 229–268. See also E. Goldschmidt, Über die numerische Verwendbarkeit der Methoden zur Auflösung unendlich vieler linearer Gleichungen, Dissertation, Würzburg, 1912, §§ 2, 5; and F. Riesz, Les Systèmes d'Équations linéaires a une Infinité d'Inconnues, 1913, p. 11, 12.

§ "On the regular and irregular solutions of some infinite systems of linear equations," Proceedings Fifth International Congress of mathematicians, vol. 1, pp. 352–366.

‖ "On non-homogeneous equations with an infinite number of variables," January, 1914, pp. 13–20.

derived conditions under which this formal expression yields a solution of the system of equations.

To the conditions (2) Helge von Koch adds

$$a_{ii} = 1, \qquad | a_{ik} | < T^{k-i},$$

and shows that

$$\overline{\lim_{i=\infty}} \sqrt[i]{c_i} < \frac{1}{k}$$

is necessary and sufficient for the existence of a unique solution $\{x_i\}$ such that

$$\overline{\lim_{i=\infty}} \sqrt[i]{x^i} < \frac{1}{k}.$$

Carmichael adds to (2) only the condition that

$$a_{ii} = 1,$$

and his sufficient conditions to be imposed on the $\{c_i\}$ are broader than those of Helge von Koch, but are more complicated in form. No information is given about the character of the solutions, and for the general system of equations there is no indication of the nature of the right-hand side for which solutions exist.

The conditions which we derive for the applicability of the Kösseritzsch method under (2) are simpler in form and more general than those of Carmichael. We also gain some information about the character of the right-hand side and the corresponding solutions for both the general and reduced systems of equations.

1. Definitions and transformation of matrices. A sequence of constants $\{c_i\}$ is of *finite norm* if $\Sigma_{i=1}^{\infty} c_i^2$ converges.

A matrix (a_{ik}) is *limited** if there exists a positive quantity M independent of $\{x_i\}$ and $\{y_i\}$ and n such that

$$\left| \sum_{i,\,k=1}^{n} a_{ik} x_i y_k \right| \leqq M \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{k=1}^{n} y_k^2}$$

for all sequences $\{x_i\}$ and $\{y_i\}$ of finite norm.

For convenience of later reference we state here some well-known properties of limited matrices.

(A) If the matrix (a_{ik}) is limited,† the transformation

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k$$

* Hellinger-Toeplitz, "Theorie der unendlichen Matrizen," Mathematische Annalen, 69, 1910, p. 296.

† Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, p. 128.

transforms every sequence $\{x_i\}$ of finite norm into a sequence $\{y_i\}$ of finite norm.

(*B*) If the matrix (a_{ik}) is limited,* and the sequences $\{x_i\}$ and $\{y_i\}$ are of finite norm,

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} a_{ik}x_iy_k = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}x_iy_k.$$

(*C*) The matrix (a_{ik}) is limited† if the sequence $\{\Sigma_{k=1}^{\infty} a_{ik}x_k\}$ is of finite norm for every sequence $\{x_i\}$ of finite norm.

For the work in the sequel we need the following lemmas concerning the transformation of matrices.

Lemma 1. *If (a_{ik}) is a matrix such that the sequence $\{a_{ik}\}$ is not of finite norm for every i, and if*

$$|\lambda_k| > \frac{\max. |a_{ik}| \quad (i = 1, 2, \cdots k)}{|\epsilon_k|}, \tag{3}$$

where $\{\epsilon_k\}$ is any sequence of finite norm, the matrix (a_{ik}/λ_k) has the property that the sequence $\{a_{ik}/\lambda_k\}$ is of finite norm for every i.

The lemma follows immediately from the term by term inequality

$$\sum_{k=1}^{\infty}\frac{a_{ik}^2}{\lambda_k^2} \leqq \sum_{k=1}^{i}\frac{a_{ik}^2}{\lambda_k^2} + \sum_{k=i+1}^{\infty}\epsilon_k^2.$$

Obviously it is sufficient to take

$$|\lambda_k| = \frac{\sqrt{\sum_{i=1}^{k} a_{ik}^2}}{|\epsilon_k|},$$

instead of the value given in (3),

Lemma 2. If *(a_{ik}) is an unlimited matrix such that $a_{ik} = 0$ for $i > k$, and if*

$$|\lambda_k| > \frac{\max. |a_{ik}| \ (i = 1, 2, \cdots k)}{|\epsilon_k|},$$

where $\{\epsilon_k\}$ is any sequence of finite norm, the matrix (a_{ik}/λ_k) is limited.

This lemma is a consequence of the inequality

$$\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}\frac{a_{ik}}{\lambda_k}x_i\right)^2 \leqq \sum_{k=1}^{\infty}\epsilon_k^2\sum_{i=1}^{k}x_i^2 \leqq \sum_{k=1}^{\infty}\epsilon_k^2\sum_{i=1}^{\infty}x_i^2,$$

and the property (*C*).

2. **Special systems of linear equations.**—Let

$$\sum_{k=1}^{\infty} a_{ik}x_k = c_i \qquad (i = 1, 2, 3, \cdots) \tag{4}$$

* Hilbert, l. c., p. 129.

† Hellinger-Toeplitz, l. c., § 10.

be a system of linear equations with the following condition on the matrix (a_{ik})

(5) $$a_{ii} \neq 0, \qquad a_{ik} = 0 \qquad i > k.$$

Let the matrix (b_{ik}) be defined by

(6) $$b_{ik} = 0 \qquad i < k,$$
$$b_{ik} = \frac{\bar{b}_{ik}}{a_{11}a_{22} \cdots a_{ii}} \qquad i \geqq k,$$

where $\bar{b}_{ik}$ is the value of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ 0 & a_{22} & \cdots & a_{2i} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & a_{ii} \end{vmatrix}$$

with $0, 0, \cdots 1$ substituted in the kth column. Then the matrix (b_{ik}) is a reciprocal matrix of (a_{ik}), or

(7) $$\Sigma a_{i\alpha} b_{k\alpha} = \begin{cases} 0 & i \neq k. \\ 1 & i = k. \end{cases}$$

THEOREM 1. *If the sequences $\{\lambda_k\}$ and $\{\mu_j\}$ are such that the sequence $\{a_{ik}/\lambda_k\}$ is of finite norm for every i, and the matrix $(\lambda_k b_{ik}/\mu_i)$ is limited, then for every sequence $\{c_i\}$ such that $\{\mu_i c_i\}$ is of finite norm, the system of equations (4) has a solution $\{x_i\}$ such that $\{\lambda_i x_i\}$ is of finite norm, and the solution is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots).$$

If the $\{a_{ik}\}$ is not of finite norm for every i, the sequence $\{\lambda_i\}$ can be chosen as in Lemma 1; if the $\{a_{ik}\}$ is of finite norm for every i, take $\lambda_k = 1$, or as a decreasing sequence in some cases. The matrix $(\lambda_i b_{ki})$ satisfies the conditions of Lemma (2), and the constants $\{\mu_i\}$ can be chosen so that the matrix $(\lambda_i b_{ki}/\mu_k)$ is limited. Applying next property (B) § 1, and the relation (7), we obtain

(8) $$\sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_{i\alpha}}{\lambda_\alpha}\left(\frac{\lambda_\alpha b_{k\alpha}}{\mu_k}\right)\mu_k c_k = \sum_{k=1}^{\infty}\sum_{\alpha=1}^{\infty} \frac{a_{i\alpha}}{\lambda_\alpha}\left(\frac{\lambda_\alpha b_{k\alpha}}{\mu_k}\right)\mu_k c_k = c_i,$$

if the sequence $\{c_i\}$ is such that $\{\mu_i c_i\}$ is of finite norm. From (8) we have that a solution of (4) is given by

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots)$$

and from property (A) § 1, the sequence $\{\lambda_i x_i\}$ is of finite norm.

Lemma (1) gives us a possible value for $\{\lambda_k\}$ in terms of the elements of the matrix (a_{ik}), if $\{a_{ik}\}$ is not of finite norm for every i, and Lemma (2) gives it for the $\{\mu_i\}$ in terms of the elements of the matrix (b_{ik}). It is desirable to have a value for $\{\mu_i\}$ in terms of a_{ik}, so that it will not be necessary to compute the b_{ik}. By the Hadamard theorem on determinants,

$$(9) \qquad |b_{ik}| \leqq \frac{\sqrt{\prod_{k=1}^{i}\sum_{i=1}^{k} a_{ik}^2}}{|a_{11}|\cdot|a_{22}|\cdots|a_{ii}|}.$$

Using this together with Lemma 2 we have

Corollary. The system of equations (4) *has a solution* $\{x_i\}$ *such that* $\{\lambda_i x_i\}$ *is of finite norm for every* $\{c_i\}$ *such that* $\{\mu_i c_i\}$ *is of finite norm, where*

$$(10) \qquad |\lambda_k| = \frac{\max. |a_{ik}|}{|\epsilon_k|} \qquad (i = 1, 2, \cdots k),$$

if $\{a_{ik}\}$ *is not of finite norm for every* i, *and*

$$(10') \qquad 0 < |\lambda_k| \leqq 1,$$

if $\{a_{ik}\}$ *is of finite norm for every* i, *and*

$$(11) \qquad |\mu_i| = \frac{\sqrt{\prod_{k=1}^{i}\sum_{j=1}^{k} a_{jk}^2}}{|a_{11}|\cdot|a_{22}|\cdots|a_{ii}|}\cdot\frac{\max. |\lambda_k|}{|\epsilon_i|} \qquad (k = 1, 2, \cdots i),$$

$\{\epsilon_i\}$ *being any sequence of finite norm.*

If $\{\lambda_i\}$ and $\{\mu_i\}$ are given by (10) and (11) respectively the sequence $\{c_i\}$ satisfying the condition of the corollary, will be such that $\Sigma_{i=1}^{\infty} |c_i|$ converges, if given by (10′) and (11) respectively, the sequence $\{c_i\}$ is of finite norm.

The same method of proof applies for the following theorem and corollary.

THEOREM 2. *If* $\{\bar{\lambda}_i\}$ *and* $\{\bar{\mu}_i\}$ *are sequences of constants such that* $\{b_{ik}/\bar{\mu}_i\}$ *is of finite norm for every* i, *and the matrix* $(\bar{\mu}_i a_{ik}/\lambda_k)$ *is a limited matrix, then any solution* $\{x_i\}$ *of the system of equations* (4) *such that* $\{\bar{\lambda}_i x_i\}$ *is of finite norm, is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots),$$

and the $\{c_i\}$ *are such that the sequence* $\{\mu_i c_i\}$ *is of finite norm.*

Corollary. If

$$(12) \qquad |\bar{\mu}_i| = \frac{\sqrt{\prod_{k=1}^{i}\sum_{i=1}^{k} a_{ik}^2}}{|a_{11}|\cdot|a_{22}|\cdots|a_{ii}|}\frac{1}{\epsilon_i},$$

and

$$|\bar{\lambda}_k| = \frac{\max. |a_{ik}|\,|\mu_i|}{|\epsilon_k|} \qquad (i = 1, 2, \cdots k)$$

where $\{\epsilon_i\}$ *is any sequence of finite norm, then any solution* $\{x_i\}$ *of the system of equations* (4) *such that the sequence* $\{\bar{\lambda}_i x_i\}$ *is of finite norm, is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots),$$

and the $\{c_i\}$ *is such that the sequence* $\{\mu_i c_i\}$ *is of finite norm.*

3. General system of equations. Let

$$\sum_{k=1}^{\infty} A_{ik} x_k = C_i \qquad (i = 1, 2, 3, \cdots) \tag{14}$$

be a system of linear equations for which the determinants

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1i} \\ A_{21} & A_{22} & \cdots & A_{2i} \\ \cdot & \cdot & \cdot & \cdot \\ A_{i1} & A_{i2} & \cdots & A_{ii} \end{vmatrix} \neq 0 \qquad (i = 1, 2, 3, \cdots).$$

Then it can be reduced to the system (4) and (5), and the relations between the matrices and right-hand sides of the equations have the form

$$A_{ik} = \sum_{k=1}^{i} \xi_{i\alpha} a_{\alpha k} \qquad i \leqq k \;\; (k = 1, 2, 3, \cdots), \tag{15}$$

$$C_i = \sum_{k=1}^{i} \xi_{i\alpha} c_\alpha \qquad (i = 1, 2, 3, \cdots). \tag{16}$$

The coefficients ξ_{ik} are not uniquely determined as each row contains a multiplicative constant. By Lemma 2 these constants can be chosen so that the matrix (ξ_{ik}) (where $\xi_{ik} = 0$ for $k > i$) is limited. Therefore by property (A) § 1, the sequence $\{C_i\}$ is of finite norm if the corresponding sequence $\{c_i\}$ is also of finite norm. From the form of (15) it follows that, if $\{a_{ik}/\lambda_k\}$ is of finite norm for every i, then $\{A_{ik}/\lambda_k\}$ is also of finite norm for every i, and vice-versa. Applying Lemma 1 and the corollary to Theorem 1, we have

THEOREM 3. *If*

$$|\lambda_i| > \frac{\text{max. } A_{ik} \;\; (i = 1, 2, \cdots k)}{\epsilon_i}$$

and

$$|\mu_i| = \frac{\sqrt{\prod_{k=1}^{i} \sum_{j=1}^{k} a_{jk}^2}}{|a_{11}| \cdot |a_{22}| \cdots |a_{ii}|} \cdot \frac{\text{max. } |\lambda_k| \; (k = 1, 2, \cdots i)}{|\epsilon_i|},$$

where $\{\epsilon_i\}$ *is any sequence of finite norm, and if* $\{C_i\}$ *has the form* (16) *where* $\{c_i\}$ *is such that* $\{\mu_i c_i\}$ *is of finite norm (and therefore* $\{C_i\}$ *is of finite norm), the system of equations* (14) *has a solution* $\{x_i\}$ *such that* $\{\lambda_i x_i\}$ *is of finite norm.*

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